Geometrical Aspects of Integrability in Nonlinear Realization Scheme

R.P.Malik [*] [†]

S. N. Bose National Centre for Basic Sciences,
Block-JD, Sector-III, Salt Lake, Calcutta-700 091, India

ABSTRACT

We discuss the integrability properties of the Boussinesq equations in the language of geometrical quantities defined on an appropriately chosen coset manifold connected with the W_3 algebra of Zamolodchikov. We provide a geometrical interpretation to the commuting conserved quantities, Lax-pair formulation, zero-curvature representation, Miura maps, etc. in the framework of nonlinear realization method.

---

[*] E-mail address: malik@boson.bose.res.in

[†] Invited talk delivered in a workshop on "Dynamical Systems: Recent Developments" (held from November 4 to November 6, 1999) at School of Physics, University of Hyderabad (India).

Two $(1 + 1)$ dimensional integrable nonlinear partial differential equations have played a notable role in the understanding of some of the physical phenomena of nature. The KdV and Boussinesq equations, their modified versions, their higher order hierarchies, etc., are the cardinal examples of such a class of equations which have found applications in as diverse areas of research as two-dimensional (2D) conformal field theories, (W)-string theories, fluid mechanics, plasma physics, 2D (W)-gravity theories, etc.[1-4]. The latter equation (i.e., the Boussinesq equation) can be realized on $u(x, t)$ and $v(x, t)$ fields as

$$\frac{\partial u}{\partial t} = -\frac{160}{3}\, v', \qquad \frac{\partial v}{\partial t} = \frac{1}{10}\, u''' - \frac{24}{5}\, u\, u', \tag{1}$$

which can be combined together to yield a nonlinear partial differential equation (NLPDE) realized on the single field $u(x, t)$ as

$$\frac{\partial^2 u}{\partial t^2} = -\frac{16}{3}\, u'''' + 256\, u\, u'' + 256\, u'\, u', \tag{2}$$

where the primes (i.e., $u' = \frac{\partial u}{\partial x}$, $v' = \frac{\partial v}{\partial x}$) denote the partial derivatives on the fields with respect to the space variable x and it has, as is physically evident, the dimension of length L (i.e. $x \sim L$). Taking into account this dimension, it can be readily seen that the evolution parameter t, fields $u(x, t)$ and $v(x, t)$ have the dimensions L^2, L^{-2} and L^{-3} respectively. In the more sophisticated language of conformal field theory, one says that the naive conformal dimensions of t, u, v are $-2, +2, +3$ respectively. In what follows, we shall be calling it the conformal spins as is the practice in the realm of research activities in the rational conformal field theories [‡]. We shall see below how this dimensional analyses will be useful in our discussions for the nonlinear realization method (connected with the group realizations on homogeneous spaces).

One of the key methods used in the discussions of the spontaneously broken gauge theories is the coset space construction (G/H) where the total Lagrangian density of the theory is found to be invariant under the group G and the vacuum of the theory is found to be invariant under the subgroup H. This coset space construction turns out to be useful in the determination of the number of Goldstone bosons, massless gauge fields, massive gauge bosons, etc., in the language of group theory. To understand the geometry behind a symmetry group G, however, the key concept is to consider it as a group of transformations acting on the coset space (G/H) for the appropriately chosen stability subgroup H. This was the starting point for the nonlinear realization method [5] applied to the spontaneously broken chiral gauge theories in the late 60's and early 70's where corresponding Lie algebras were found to be *linear*. In the recent past, however, this method was exploited for the discussion of geometry behind the 2D (super) NLPDE connected with (super) W-type algebras [6,7] which are *nonlinear* to begin with. In fact, in these works, an infinite dimensional *linear* algebra was constructed from the nonlinear (super)

[‡] 2D conformal field theory can be written in terms of the complex variables z and $\bar{z}$. There are holomorphic and antiholomorphic parts in the theoy which factorize due to conformal invariance. The conformal spin is actually defined as $(h - \bar{h})$ where h and $\bar{h}$ are the conformal weights of a primary field.

W_3 algebra and all the techniques of the nonlinear realization method were exploited. In this presentation, we shall see how some of the key properties of integrability of the Boussinesq equation can be understood in the language of geometry on the coset manifold.

Before we come over to the description of the coset space construction for the W_3 algebra of Zamolodchikov, we shall dwell a bit on the basic concepts associated with this method. A realization of a Lie group (or corresponding Lie algebra) is a mathematical concept and it corresponds to the validity of certain specific type of differential equations for a group valued function associated with the Lie group. As an example, it is a well known fact that the *linear* realization of a compact, connected (semi)simple Lie group (or corresponding Lie algebra) is nothing but the (matrix) representation of the Lie algebra. To elaborate and explain these statements, let us begin with a set of fields ψ_n (where n is the multiplicity) and consider the action of the group elements g of the compact Lie group G such that the field ψ_n transforms as [8, 9]

$$g : \quad \psi_n \quad \rightarrow \quad f_n(\psi, g), \tag{3}$$

where $f_n(\psi, g)$ is an abstract form of the transformed field ψ_n. Now one can impose all the four properties of a group on transformation (3). For instance, the action of the identity $\mathbf{1}$, the existence of the inverse ($g^{-1}g = gg^{-1} = \mathbf{1}$) and the closure property can be explicitly explained in the language of transformations as

$$\mathbf{1} : \quad \psi_n \quad \rightarrow \quad f_n(\psi, \mathbf{1}) = \psi_n, \tag{4}$$

$$g \cdot g^{-1} : \quad \psi_n \rightarrow f_n(\psi, g^{-1}) \rightarrow f_n(f(\psi, g^{-1}); g) \equiv \psi_n, \tag{5}$$

$$g^{-1} \cdot g : \quad \psi_n \rightarrow f_n(\psi, g) \rightarrow f_n(f(\psi, g); g^{-1}) \equiv \psi_n, \tag{6}$$

$$g_1 \cdot g_2 : \psi_n \rightarrow f_n(\psi, g_2) \rightarrow f_n(f(\psi, g_2); g_1) \equiv f_n(\psi, g_1 \cdot g_2), \tag{7}$$

where the last equation is just the closure property under binary operation ($\cdot$). Here the binary operation is nothing but the transformation (3). One can exploit this property to establish the associativity:

$$f_n\Big(\psi, g_1 \cdot (g_2 \cdot g_3)\Big) = f_n\Big(\psi, (g_1 \cdot g_2) \cdot g_3\Big). \tag{8}$$

So far, the group element g was treated as an abstract object, now we can write the explicit form of it in terms of a set of infinitesimal transformation parameters ϵ_α as

$$g = \mathbf{1} + i\epsilon_\alpha \Gamma_\alpha, \tag{9}$$

where Γ_α are the set of generators for the above transformations which obey a commutation relationship for the given Lie algebra as

$$[\Gamma_\alpha, \Gamma_\beta] = iC_{\alpha\beta\gamma}\Gamma_\gamma. \tag{10}$$

Here $C_{\alpha\beta\gamma}$ are the structure constants of the algebra. The explicit form of the transformed field (with $f_n(\psi, \mathbf{1}) = \psi_n$, and eqn. (9)) is

$$
\begin{aligned}
f_n(\psi, g) &= f_n(\psi, \mathbf{1} + i\epsilon_\alpha \Gamma_\alpha) \\
&\equiv \psi_n + i\epsilon_\alpha f_{n\alpha}(\psi) + O(\epsilon^2),
\end{aligned}
\tag{11}
$$

where $f_{n\alpha}(\psi)$ is a group valued function and it retains the information about the multiplicity as well as the group properties. For a given Lie algebra, the group valued functions $f_{n\alpha}$ can *not* be chosen in a perfectly arbitrary way. Rather, they obey certain specific kind of differential equation for the given Lie algebra. For instance, using the closure property of equation (7), it can be seen that

$$
\begin{aligned}
f_n(\psi, g_1^{-1} \cdot g_2 \cdot g_1) &= f_n(f(\psi, g_2 \cdot g_1); g_1^{-1}) \\
&\equiv f_n(f\{f(\psi, g_1); g_2\}; g_1^{-1}).
\end{aligned}
\tag{12}
$$

With the following inputs from the group properties (see, e.g., eqns. 4–8)

$$
f_n(\psi_m, \mathbf{1}) = \delta_{nm}\psi_m, \quad f_n(f(\psi, g); g^{-1}) = \psi_n,
\tag{13a}
$$

$$
f_n(f(\psi, g)) = \psi_n + i\epsilon_\alpha \left(\tfrac{\partial f_n(\psi)}{\partial \psi_m}\right) f_{m\alpha} + O(\epsilon^2),
\tag{13b}
$$

we obtain the following differential equation satisfied by $f_{n\alpha}$

$$
f_{n\beta}\, C_{\alpha\gamma\beta} = -i\, \tfrac{\partial f_{n\alpha}}{\partial \psi_m}\, f_{m\gamma} + i\, \tfrac{\partial f_{n\gamma}}{\partial \psi_m}\, f_{m\alpha}.
\tag{14}
$$

All set of functions $f_{n\alpha}(\psi)$ that satisfy the above differential equation is said to provide a *realization* of the given Lie algebra. For instance, it can be seen that the following *linear* choice of $f_{n\alpha}(\psi)$

$$
f_{n\alpha}(\psi) = (\tau_\alpha)_{nm}\, \psi_m,
\tag{15}
$$

leads to

$$
[\tau_\alpha, \tau_\beta] = iC_{\alpha\beta\gamma}\, \tau_\gamma,
\tag{16}
$$

which is nothing but the matrix representation of the Lie algebra in (10). For the spontaneously broken (chiral) gauge theories, the generators Γ_α of the full group G have two parts. Some of the generators T_i (i : dimensionality of H) belong to the subgroup H and the rest of the generators X_a belong to the coset space G/H. Thus, one can parametrize the coset space by coset fields and the generators X_a. It was shown in Refs. [5] that one can parametrize the coset space in terms of *exponentials* and it *also* provides a *realization* for the factorized Lie algebra under consideration. Due to the presence of exponentials, this special realization is known as the *nonlinear realization*. It can be seen that the *linear* term of this realization (exponentials) is nothing but the representaion (cf. eqn. (16)) of the Lie algebra *if the stability subalgebra contains only identity element of the Lie algebra*. Thus, we see that the linear realization is a special case of the nonlinear realization.

The first thing we note, before the application of nonlinear realization method to the classical W_3 algebra with central extension (parametrized by the central charge c)

$$
\begin{aligned}
&[L_n, L_m] = (n-m)L_{n+m} + \tfrac{c}{12}\,(n^3 - n)\,\delta_{n+m,0}, \quad (-\infty \leq n, m \leq +\infty) \\
&[L_n, W_m] = (2n-m)\,W_{n+m}, \\
&[W_n, W_m] = 16\,(n-m)\,\Lambda_{n+m} - \tfrac{c}{9}(n^3-n)(n^2-4)\,\delta_{n+m,0} \\
&\qquad\qquad -\tfrac{8}{3}\,(n-m)(n^2+m^2 - \tfrac{1}{2}nm - 4)\,L_{n+m},
\end{aligned} \tag{17}
$$

is the fact that it is a *nonlinear* algebra because of the composite and nonlinear nature of $\Lambda_n = -\tfrac{8}{3}\sum_m L_{n-m}L_m$. It was thus essential to get a *linear* algebra out of it so that the whole arsenal of techniques of coset space construction can be applied here. To this goal, in Ref. [6], all the higher spin composite generators were treated as independent generators. Invoking this idea immediately entails upon the W_3 algebra to become an infinite dimensional linear algebra W_3^∞ as

$$
W_3^\infty = \{L_n, W_n, \Lambda_n, \Phi_n.........J_n^h...........\}, \tag{18}
$$

where $\Phi_n \equiv (WT)_n$ is the conformal spin-5 composite generator and J_n^h is a generic composite generator with conformal spin-h. It can be readily seen by taking a single contraction of the OPE's that all the higher conformal spin (≥ 4) composite generators form a closed algebra among themselves and they form an ideal. One of the key subalgebras of this infinite dimensional algebra is the one in which the Laurent indices of the generators with conformal spin-h (i.e., $J(z) = \sum_n J_n^h\, z^{-n-h}$) vary from $-(h-1)$ to ∞. For instance, for the conformal spin-2, we have the generators: $L_{-1}, L_0, L_{+1}, L_{+2}.......$ and for the conformal spin-3, we have $W_{-2}, W_{-1}, W_0, W_1.........$ and so on and so forth. The stability subalgebra $\mathcal{H}$ of our interest, from this truncated version of W_3^∞, is

$$
\mathcal{H} = \{W_{-1} + 2L_{-1}, W_0, W_1, W_2, L_1, L_0, \Lambda_n(n \geq -3), J_n^h(h \geq 5, n \geq -h+1)\}, \tag{19}
$$

where it can be easily seen that W_{-2} is not present and L_{-1} appears in a particular linear combination with W_{-1} for the closure of the algebra. Thus, W_{-2} and L_{-1} can be taken into the coset space. It will be also noticed that all the higher order conformal spin composite generators have been taken into the stability subalgebra as they form an ideal. Now the element g in the coset space can be parametrized as

$$
g \in \frac{\mathcal{G}}{\mathcal{H}} = e^{tW_{-2}}e^{xL_{-1}}e^{\psi_3 L_3}\Big(\Pi_{n=4}e^{\psi_n L_n}e^{\xi_n W_n}\Big)e^{uL_2}e^{vW_3}. \tag{20}
$$

It is an interesting point to note that out of all the generators in the coset space, only two generators (i.e., L_{-1}, W_{-2}) commute with each other. They have the dimensions of length as : $W_{-2} \sim L^{-2}, L_{-1} \sim L^{-1}$. To make the exponentials in equation (15) dimensionless, it is clear that t and x must have dimensions of length as: $t \sim L^2$, $x \sim L$. It is illuminating to see that these are exactly the dimensions of t and x for the Boussinesq equations which were discussed after eqn.(2). The commutativity of the generators W_{-2}, L_{-1} ensures that the t and x directions are linearly independent on the coset manifold and, therefore, they

5

can be treated as coordinates. This feature should be contrasted with other generators and associated tower of coset (Goldstone) fields $u, v, \psi_3, \xi_4, \psi_4, \xi_5$.......... which cannot be treated as coordinates. Now any point on the manifold can be parametrized by the coordinates x and t and all the fields can be treated as functions of these coordinates.

The most important geometrical quantity in the framework of nonlinear realization method is the one-differential Cartan form $\Omega = g^{-1}dg$ (where $g \in \frac{\mathcal{G}}{\mathcal{H}}$) in terms of which the curvature tensor, torsion, complex structure, etc. of the coset manifold can be determined. Due to its very structure, it obeys the following Maurer-Cartan equation

$$d^{ext}\, \Omega + \Omega \wedge \Omega = 0, \tag{21}$$

which is nothing but *the zero-curvature representation for the non-Abelain gauge theory if we choose the 1-differential Cartan form in terms of the gauge connections* $(A_\mu = A_\mu^a T^a)$ *as:* $\Omega = A_\mu dx^\mu$. Here T^a are the generators of the Lie algebra under consideration. Now, it can be seen that the non-Abelain curvature tensor, emerging from (21), is zero, namely;

$$\begin{aligned} F_{\mu\nu} &= [D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = 0, \\ D_\mu &= \partial_\mu + A_\mu, \qquad A_\mu = A_\mu^a T^a\, dx^\mu. \end{aligned} \tag{22}$$

In fact, in the language of gauge theory, the choice $\Omega = g^{-1}dg$ is eaxctly like the *pure gauge* choice. Thus, the zero curvature representation is bound to be satisfied. For the truncated version of the algebra W_3^∞, we obtain

$$\Omega = g^{-1}dg = \sum_{n=-1}^{\infty}\omega_n L_n + \sum_{n=-2}^{\infty}\theta_n W_n + \text{ higher spin contributions.} \tag{23}$$

As higher spin composite generators form an ideal, it is essential to know only some of the lower order forms to obtain the dynamical equations of motion if we exploit the ideas of Inverse Higgs-Covariant Reduction (IH-CR) procedure [10,11]. These lower order forms are

$$\begin{aligned} \omega_{-1} &= dx, \quad \omega_0 = 0, \quad \omega_1 = -3udx + 160vdt, \quad \omega_2 = du - 4\psi_3 dx + 320\xi_4 dt, \\ \omega_3 &= d\psi_3 + \left(\tfrac{3}{2}u^2 - 5\psi_4\right)dx + \left(560\xi_5 - 240uv\right)dt, \\ \omega_4 &= d\psi_4 - 6\psi_5 dx + \left(896\xi_6 - 192v\psi_3 - 768u\xi_4\right)dt, \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta_{-2} &= dt, \quad \theta_{-1} = 0, \quad \theta_0 = -6udt, \quad \theta_1 = -8\psi_3 dt, \\ \theta_2 &= -5v\, dx + \left(12u^2 - 10\psi_4\right)dt, \qquad \theta_3 = dv - 6\xi_4 dx + \left(24u\psi_3 - 12\psi_5\right)dt \\ &\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{24}$$

According to the IH-CR procedure, one can set equal to zero all the components of the forms connected to the generators in the coset space. In fact, these forms transform homogeneously under the left action of the truncated version of W_3^∞ symmetry and setting them equal to zero does not spoil the symmetry of the group. These constraints are just like the "gauge-fixing" conditions on the gauge-connections in the language of gauge theory. To make this statement more transparent, it can be seen that the following constraints:

$$\omega_n = 0, \quad \forall \quad n \geq 2, \qquad \theta_n = 0, \quad \forall \quad n \geq 3, \tag{25}$$

lead to the following kinematical and dynamical equations

$$
\begin{aligned}
\psi_3 &= \tfrac{1}{4}\, u', & \psi_4 &= \tfrac{1}{5}\psi_3' + \tfrac{3}{10}\, u^2, & \psi_5 &= \tfrac{1}{6}\, \psi_4', & \xi_4 &= \tfrac{1}{6}\, v', \\
\dot{u} &= -\tfrac{160}{3}v', & \dot{v} &= \tfrac{1}{10}\, u''' - \tfrac{24}{5}u\, u', & \dot{u} &= \tfrac{\partial u}{\partial t}, & \dot{v} &= \tfrac{\partial v}{\partial t}.
\end{aligned}
\tag{26}
$$

Thus, we see that the Boussinesq equation of eqn. (1) emerges here by the IH-CR procedure applied on the coset manifold as all the tower of fields can be expressed in terms of the essential fields u and v and the derivatives on them . In the language of geometrical properties on the coset manifold, *it can be seen that the Boussinesq equations are nothing but the embedding conditions on a two dimensional (x,t) geodesic surface (parametrized by the basic fields $u(x,t)$ and $v(x,t)$ and the derivatives on them) when one singles out this hypersurface from the infinite dimensional coset manifold.*

Mathematically, the Boussinesq equation can be understood in the language of group motion when infinite dimensional algebra of the infinite dimensional coset manifold reduces covariantly to the 'covariant reduced algebra' generated by the elements of the $sl(3,R)$ algebra. In other words, the original Cartan form now reduces to a reduced Cartan form (due to IH-CR procedure) as

$$
\Omega = g^{-1}dg \rightarrow \Omega_{red} = g_{red}^{-1}\, d\, g_{red} = \sum_{n=-2}^{n=2}\theta_n W_n + \sum_{n=-1}^{n=+1}\omega_n L_n,
\tag{27}
$$

which satisfies the Maurer-Cartan equation: $d^{ext}\Omega_{red} + \Omega_{red} \wedge \Omega_{red} = 0$. The explicit form of this reduced Cartan form is

$$
\begin{aligned}
\Omega_{red} &= A_x\, dx + A_t\, dt, \\
A_x &= L_{-1} - 3uL_1 - 5vW_2, \\
A_t &= 160vL_1 - 6uW_0 + W_{-2} - 8\psi_3 W_1 + (12u^2 - 10\psi_4)W_2.
\end{aligned}
\tag{28}
$$

It can be now readily seen that the following zero-curvature condition

$$
F_{tx} = [\partial_t + A_t, \partial_x + A_x] = 0,
\tag{29}
$$

leads to the derivation of the Boussinesq equations. It will be noticed that A_x and A_t are nothing but the $sl(3,R)$ valued *Drinfeld-Sokolov type Lax-pairs* in eqn. (28). *In the language of geometry, these Lax-pairs can be understood as the projections of the reduced Cartan form along x and t directions of the coset manifold* (cf. eqn. (28)). It is straightforward to notice that if the generators W_2, W_1, L_1 are taken out from the stability subalgebra in (19), still the algebra will be closed. Thus, we obtain a new subalgebra $\mathcal{H}_1$ from $\mathcal{H}$ as

$$
\mathcal{H}_1 = \{W_{-1} + 2L_{-1}, W_0, L_0, \Lambda_n(n \geq -3), J_n^h(h \geq 5, n \geq -h+1)\}.
\tag{30}
$$

In this case, the coset space can be parametrized as

$$
g_1 \in \frac{\mathcal{G}}{\mathcal{H}_1} = e^{tW_{-2}}e^{xL_{-1}}e^{\psi_3 L_3}\left(\Pi_{n=4}e^{\psi_n L_n}e^{\xi_n W_n}\right)e^{uL_2}e^{vW_3}e^{u_1 L_1}e^{v_1 W_1}e^{v_2 W_2}.
\tag{31}
$$

Furthermore, one can take out W_0, L_0 from the stability subalgebra (30) and still algebra will be closed with only one $U(1)$ basic generator $(W_{-1} + 2L_{-1})$. Now the coset element is

$$g_2 \in \frac{\mathcal{G}}{\mathcal{H}_2} = g_1 \, e^{u_0 L_0} e^{v_0 W_0}, \tag{32}$$

where $\mathcal{H}_2$ is given by

$$\mathcal{H}_2 = \{W_{-1} + 2L_{-1}, \Lambda_n(n \geq -3), J_n^h(h \geq 5, n \geq -h + 1)\}. \tag{33}$$

In both the cases of g_1 and g_2, one can define a one-differential Cartan form $\Omega_1 = g_1^{-1} \, d \, g_1$ and $\Omega_2 = g_2^{-1} \, d \, g_2$ and apply the IH-CR procedure [10, 11] on it. This leads to a covariant relationship between essential fields u_1, v_1 of the coset manifold (31) and u, v fields of coset the manifold (20). Similarly, one gets a relationship between essential fields u_0, v_0 of coset manifold (32) and u_1, v_1 fields of (30). These are nothing but the so-called Miura maps. The dynamical equations on u_0, v_0 and u_1, v_1 fields can also be obtained due to appropriate application of CR procedure on coset manifolds (30) and (32) as we obtained for the essential fields u, v in eqn. (26). Thus, we see that *the Miura maps are nothing but the kinematical relationships among essential fields when one goes covariantly from one coset manifold to another one.*

It is obvious that the kinematical and dynamical relationships can be obtained from the nonlinear realization method by application of the IH-CR procedure [10, 11]. For the first time, however, this procedure was extended one step further to derive commuting conserved quantities for the Boussinesq equations in Ref. [12]. We shall briefly dwell a bit on it. For the derivation of the commuting conserved quantities, one has to compute more higher order forms than the ones required for the derivation of the dynamical equations. Some of these forms are

$$
\begin{aligned}
\omega_5 &= d\psi_5 + u d\psi_3 + (\tfrac{1}{2}u^3 - 5u\psi_4 + 2\psi_3^2 - 40v^2 - 7\psi_6) \, dx \\
&+ (192u^2 v - 336u\xi_5 - 704\psi_3\xi_4 - 160v\psi_4 + 1344\xi_7) \, dt, \\
\omega_6 &= d\psi_6 + 2u d\psi_4 + (8\psi_3\psi_4 - 12u\psi_5 - 8\psi_7) \, dx \\
&+ (1920\xi_8 + 768u\xi_6 + 768u^2\xi_4 - 640\psi_4\xi_4 - 1664\psi_3\xi_5) \, dt, \\
. & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
. & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\theta_4 &= d\xi_4 + (3uv - 7\xi_5) \, dx + (20\psi_3^2 + 20u\psi_4 - 14\psi_6 - 8u^3 - 80v^2) \, dt, \\
\theta_5 &= d\xi_5 + v du + (6u\xi_4 - 4v\psi_3 - 8\xi_6) dx \\
&+ (56\psi_3\psi_4 + 320v\xi_4 + 12u\psi_5 - 12u^2\psi_3 - 16\psi_7) \, dt, \\
\theta_6 &= d\xi_6 + 3v d\psi_3 + (\tfrac{9}{2}u^2 v - 15v\psi_4 - 9\xi_7) \, dx \\
&+ bigl(72\psi_3\psi_5 + 1680v\xi_5 + 30\psi_4^2 - 360v^2 u - 18\psi_8) \, dt, \\
. & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
. & \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{34}
$$

It can be readily seen that, due to IH-CR procedure, we can set the dt projection of the form θ_5 equal to zero. This leads to the following equation

$$\frac{\partial \xi_5}{\partial t} = 16\psi_7 + 12u^2\psi_3 - 12u\psi_5 - 56\psi_3\psi_4 - 320v\xi_4. \tag{35}$$

8

Now using the kinematical relationships, it can be seen that the r.h.s. of the above expression is a total space derivative

$$\frac{\partial \xi_5}{\partial t} = \frac{\partial}{\partial x} \left[2\psi_6 + \frac{1}{5}u^3 - 2u\psi_4 - \frac{16}{3}\psi_3^2 \right], \tag{36}$$

which ultimately leads to the following conservation law

$$\frac{\partial (uv)}{\partial t} = \frac{\partial}{\partial x} \left[-\frac{11}{5}u^3 + 2u\psi_4 - \frac{16}{3}\psi_3^2 - \frac{80}{3} v^2 \right], \tag{37}$$

where we have used $\xi_5 = \frac{\xi_4'}{7} + \frac{3}{7}uv, \psi_6 = \frac{1}{7}(\psi_5' + 2\psi_3^2 - u^3 - 40v^2)$ from the kinematical relationships. Similarly other conserved quantities can be calculated (see, e.g., Ref. [12] for details). Some of the conserved quantities are

$$
\begin{aligned}
H_1 &= \frac{c}{2} \int dx\, u(x,t), \quad H_2 = -\frac{40c}{3} \int dx\, v(x,t), \quad H_4 = c \int dx\, (uv)(x,t), \\
H_5 &= -c \int dx\, \left[\frac{(u')^2}{20} + \frac{4u^3}{5} + \frac{80v^2}{3} \right], \\
H_7 &= c \int dx\, \left[\frac{(u'')^2}{3200} + \frac{9u(u')^2}{400} + \frac{(v')^2}{6} + \frac{3u^4}{50} + 4uv^2 \right], \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots \\
&\quad \dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
\tag{38}
$$

Here the subscripts for the conserved quantities stand for the naive conformal dimensions (i.e. $H_1 \sim L^{-1}, H_2 \sim L^{-2}$ etc.). The commutativity of the conserved quantities (i.e., $\{H_i, H_j\} = 0, \quad i,j = 1,2,4,5,7.....$) can be established if we exploit the following second Hamiltonian structure associated with u and v fields for the classical W_3 Poisson brackets

$$
\begin{aligned}
\{u(x,t), u(y,t)\} &= \frac{2}{c} \left[\frac{1}{6}\frac{\partial^3}{\partial y^3} - 2u(y)\frac{\partial}{\partial y} - \frac{\partial u}{\partial y} \right] \delta(x-y), \\
\{u(x,t), v(x,t)\} &= -\frac{2}{c} \left[3v(y)\frac{\partial}{\partial y} + \frac{\partial v}{\partial y} \right] \delta(x-y), \\
\{v(x,t), v(y,t)\} &= \frac{3}{100c} \left[-\frac{1}{48}\frac{\partial^5}{\partial y^5} + \frac{5}{4}u(y)\frac{\partial^3}{\partial y^3} + \frac{15}{8}\frac{\partial u}{\partial y}\frac{\partial^2}{\partial y^2} \right. \\
&\quad + \left. (\frac{9}{8}\frac{\partial^2 u}{\partial y^2} - 12u^2)\frac{\partial}{\partial y} + (\frac{1}{4}\frac{\partial^3 u}{\partial y^3} - 12u\frac{\partial u}{\partial y}) \right]\delta(x-y).
\end{aligned}
\tag{39}
$$

It will be noticed that there are no conserved quantities for the conformal dimensions 3, 6, 9, 12.....(n = 0 mod 3). This is in agreement with the famous Lenard recursion relations for the commuting conserved quantities for the Boussinesq equations. These conserved quantities can be understood in terms of the generators of the infinite dimensional algebra W_3^∞. For instance, if we take the Laurent mode decomposition for the u and v fields and consider the holomorphic and antiholomorphic parts together, the contour integration in (38) will lead to the following set of generators modulo some constant factors:

$$\{L_{-1}, W_{-2}, \Phi_{-4}, S_{-5}................\} \tag{40}$$

where $\Phi = \frac{48}{c}(TW), S = \frac{1}{c} (W^2 - \frac{128}{3c}T^3 + \frac{4}{3} (\partial T)^2)$.......etc. These generators form a *Cartan subalgebra* in the infinite dimensional algebra W_3^∞ as they commute among themselves. In the framework of nonlinear realization method, these generators correspond to the translation generators on the infinite dimensional coset manifold. For instance, the first

conserved quantity H_1 corresponds to the generator L_{-1} which is nothing but the space translation (momentum) generator. The rest of the conserved quantities correspond to the time evolution generators on the coset manifold. Their commutativity corresponds to the linear independence of the directions of the space x and all the other "time" evolution parameters with one another.

To summarize, we have shown that: (i) the Boussinesq equations are the embedding conditions on a 2D geodesic hypersurface in the infinite dimensional coset manifold, (ii) Miura maps are the covariant kinematical relationships among the essential fields as one goes covariantly from one coset manifold to another, (iii) Lax-pairs are the components of projections of the reduced Cartan forms along x and t directions of the coset manifold, (iv) Commuting conserved quantities are the translation generators on the manifold and they form a Cartan subalgebra in the infinite dimensional algebra W_3^∞, (v) Commutativity of the conserved quantities are reflected in the linear independence of all the evolution directions $(x, t, t'.....)$ on the coset manifold.

References

[1] A. M. Polyakov, Mod. Phys. Lett. **A 2**, 893 (1987), V. Z. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. **A 3**, 819 (1988).

[2] A. Das, W. -J. Huang and S. Roy, Int. J. Mod. Phys. **A 7**, 3447 (1992), S. Ghosh and S. Roy, Int. J. Mod. Phys. **A 8**, 895 (1993).

[3] J. -L. Gervais, Phys. Lett. **B 160**, 277 (1985),
I. Bakas, Nucl. Phys. **B 302**, 189 (1988).

[4] V. Drinfeld and V. Sokolov, J. Sov. Math. **30**, 1975 (1985),
A. Bilal and J. -L. Gervais, Nucl. Phys. **B 314**, 579 (1989).

[5] S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2239 (1969), C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. **177**, 2247 (1969); D. V. Volkov, Sov. J. Part. Nucl. **4**, 3 (1973).

[6] E. Ivanov, S. Krivonos and R. P. Malik, Int. J. Mod. Phys. **A 8**, 3199 (1993).

[7] E. Ivanov, S. Krivonos and R. P. Malik, Int. J. Mod. Phys. **A 10**, 253 (1995).

[8] V. I. Ogievetsky, in *Proceedings of the Xth- Winter School of Theoretical Physics*, Wroclaw (Poland), Vol. **1**, 117 (1974).

[9] S. Weinberg, *The quantum theory of fields: Modern Applications*, Vol. II, Chapter 19, (Cambrdige Univ. Press: 1996).

[10] E. Ivanov and V. Ogievetsky, Teor. Math. Fiz. **25**, 164 (1975).

[11] E. Ivanov and S. Krivonos, Teor. Math. Fiz. **58**, 200 (1984), Lett. Math. Phys. **8**, 39 (1984).

[12] R. P. Malik, Int. J. Mod. Phys. **A 12** , 231 (1997).
See also, e.g., *Commuting conserved quantities in nonlinear realizations of W_3*, JINR-preprint: E2-96-120, (1996) (unpublished).